

The Cooperon and the Random Matrix Model for Type-II Superconductors

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We derive the connection between the Cooperon problem in weak localization theory and the random matrix description of type-II superconductors. As magnetic field and disorder increase, an extreme type-II superconductor crosses over from a state in which the low energy quasiparticles are primarily localized and the density of states is determined by the electronic structure of individual vortices, to a ‘chaotic’ state, in which quasiparticles are primarily extended and the density of states is determined by the random matrix description.

PACS numbers: 74.60.-w, 74.25.Jb, 74.50.+r, 05.40.+j

I. INTRODUCTION

In this paper we explore the hypothesis that electrons in a superconductor with explicit time-reversal breaking pair via random matrix elements. This hypothesis combines ideas and techniques from three fields of study usually considered unrelated: random matrices, type-II superconductors, and the Cooperon problem of weak localization. We will show that incorporating these ideas into a random matrix model allows us to calculate the density of states of a type-II superconductor non-perturbatively and leads to new possibilities for investigating chaotic behavior in complex quantum systems.

This section contains brief introductions, with references to extensive reviews, of the main ingredients of the theory. In section II we describe the random matrix model, a generalization of the Anderson description of disordered superconductors to superconductors in a magnetic field. In section III we derive the connection between this description of type-II superconductors and the Cooperon. In section IV we show that this connection yields the correct answer for the semiclassical upper critical field, H_{c2} . In section V we discuss the weak field limit and the relation between the random matrix model and the Abrikosov-Gor’kov theory. Finally, in section VI we summarize the principal results and describe open questions and further possible areas of research.

A. Random Matrices

Random matrices were introduced into theoretical physics by Wigner in the 1950’s as a tool for understanding the distribution of energy level spacings and widths observed in nuclei [1]. The distribution of level spacings in nuclei was compared with the distribution of level spacings of a large matrix filled with random (uncorrelated) matrix elements; quantitatively good agreement was eventually found [2]. The idea of comparing the level spacings of complex quantum systems with the level spacings of random matrices was eventually extended

to atoms [3], small metallic particles [4], and classically chaotic systems [5]. In all of these applications, in condensed matter, atomic, and nuclear physics, the focus has been on understanding the distribution of the *spacings* of energy levels, that is, the probability $P(E, x)dx$ that the neighbor of a level at energy E has an energy in the range $E + x$ to $E + x + dx$. In contrast, we will use random matrices to understand the *density* of energy levels, that is, the number of energy levels $\rho(E)dE$ in the range E to $E + dE$.

Random matrices have also been studied in the context of field theory, spurred by ’t Hooft and Brezin et al.’s discovery of the connection with the large- N limit of certain $SU(N)$ -invariant field theories [6]. A connection between the Feynmann diagram expansion for random matrix theories and the sum over random surfaces in two-dimensional quantum gravity was also established [7]. More recently, random matrices have resurfaced as a possible unification of various string theories [8], and as a concrete example of a more general theory of “non-commuting random variables” [9].

Extensive recent reviews can be found in Refs. [10] and [11].

B. Type-II Superconductors

The BCS theory [12] describes the electronic properties of ideal, uniform superconductors; it applies to the interiors of type-I superconductors, which expel magnetic field, or to type-II superconductors below the field of first penetration, H_{c1} . The presence of a magnetic field, which occurs for type-II superconductors in the mixed state, $H_{c1} < H < H_{c2}$, makes the calculation of electronic properties significantly more complicated. The magnetic field destroys both the spatial uniformity and the time-reversal symmetry implicit in the BCS description.

Magnetic field penetrates a superconductor in the form of tubes of magnetic flux (vortices) enclosing circulating quasiparticle currents. These currents lead to electronic states which are bound to individual vortices. Such states are not only spatially inhomogeneous, but may

overlap strongly with neighboring vortices and form energy bands, the details of which will be sensitive to the presence of disorder or the deviations from an ideal vortex lattice that are found in most materials.

It is possible to attack the problem of the electronic properties of type-II superconductors numerically. For a fixed type of vortex lattice, a large unit cell can be considered which contains one or more vortices. Either the full Bogoliubov-de Gennes equations [13] or the approximate Eilenberger equations [14] (derived by assuming slow variations on the scale of the Fermi wavelength) may be solved to obtain the quasiparticle eigenstates [15,16].

The numerical approach to the electronic properties of type-II superconductors has several drawbacks: (1) It is difficult to get useful results numerically, both because of the large number of electrons per unit cell for realistic materials and because of the widely different energy scales in the problem. (The cyclotron frequency is much less than the BCS gap, which is in turn much less than the Debye energy.) (2) In a magnetic field, unlike conventional superconductors at zero field, the electronic properties are sensitive to disorder. The highly idealized impurity-free perfect vortex lattice is unrealistic for most materials. (3) The numerical approach calculates every quasiparticle wavefunction in the superconductor. This is often an overkill: for many applications we need much less information, for example, just the single-particle density of states. An approach which is simpler and more general than numerically solving the Bogoliubov-de Gennes equations is desirable.

Analytically manageable extensions of the BCS theory to disordered superconductors were given both by Anderson [17] and by Abrikosov and Gor'kov [18]. Anderson generalized the BCS description to show that non-magnetic impurities have little effect on conventional superconductors. Abrikosov and Gor'kov introduced a diagrammatic technique to include the effects of a random ensemble of impurities, and showed that magnetic impurities can suppress superconductivity.

Neither the Anderson nor the Abrikosov-Gor'kov description, however, applies directly to the mixed state of type-II superconductors. The Abrikosov-Gor'kov description requires spatial homogeneity, which is violated in the mixed state, and, in order to apply to a type-II superconductor, must assume that random magnetic impurities are equivalent to an applied magnetic field [19]. The Anderson description is more general, in the sense that it is not tied to the assumption of averaging over a random ensemble of impurities, however it requires both time-reversal symmetry and spatial homogeneity. The method described below can be considered a generalization of the Anderson description which has neither of these constraints.

C. The Cooperon

The problem of the localization of a quantum particle by a random potential was originally posed by Anderson in 1958 [20], in the context of understanding metal-insulator transitions, and has led to a large body of work on the subject of the quantum effects of disorder in metals. The field has been summarized in a number of recent reviews [21–24].

One of the theoretical approaches to understanding the quantum effects of disorder is impurity-averaged perturbation theory [18,25]. For non-interacting electrons scattered by rigid impurities, Green's functions may be expanded perturbatively in the impurity potential. Averaging over a random ensemble of impurity potentials yields classes of diagrams which can be organized by the powers of $k_F \ell$ they contain, where k_F is the Fermi momentum and ℓ is the mean free path. Diagrams within a given class may be summed, resulting in a series expansion in $k_F \ell$ for the desired Green's function.

Impurity-averaged perturbation theory can be used to calculate the conductivity of an electron gas in the presence of random disorder. The conductivity is related to the current-current response function by the Kubo-Greenwood formula, and the impurity diagrams for the current-current response function can be summed. It was recognized early on that a certain class of diagrams, the maximally crossed ones, yield a significant quantum correction to the classical conductivity [26]. This estimate for the quantum correction was used as support for the scaling theory of localization [27,28].

The maximally crossed diagrams for the current-current response function (a particle-hole response function) can be related by time-reversal to ladder diagrams for a response function in the particle-particle channel. The impurity-averaged particle-particle response function was called the 'Cooperon' by Altshuler *et al.* [29,30]. It is this response function, the Cooperon, originally used to study the conductivity of a disordered electron gas, which appears in the random matrix description of type-II superconductors described below.

II. RANDOM MATRIX MODEL

The starting point for an electronic description of type-II superconductors is the Hamiltonian [31]

$$\mathcal{H} = \int d\mathbf{r} c_{\mathbf{r}\sigma}^\dagger \mathcal{H}_0(\mathbf{r}) c_{\mathbf{r}\sigma} - \frac{1}{2} V_0 \Omega \int d\mathbf{r} c_{\mathbf{r}\sigma}^\dagger c_{\mathbf{r}\sigma'}^\dagger c_{\mathbf{r}\sigma'} c_{\mathbf{r}\sigma}. \quad (1)$$

Here Ω is the system volume, $V_0 > 0$ represents an attractive short-ranged interaction, the creation operators satisfy

$$\begin{aligned}\{c_{\mathbf{r}\sigma}, c_{\mathbf{r}'\sigma'}^\dagger\} &= \delta_{\sigma\sigma'} \delta^3(\mathbf{r} - \mathbf{r}') \\ \{c_{\mathbf{r}\sigma}, c_{\mathbf{r}'\sigma'}\} &= \{c_{\mathbf{r}\sigma}^\dagger, c_{\mathbf{r}'\sigma'}^\dagger\} = 0,\end{aligned}$$

and the bare Hamiltonian \mathcal{H}_0 is

$$\mathcal{H}_0(\mathbf{r}) \equiv \frac{1}{2m} \left(i\nabla - \frac{e}{c} \mathbf{A}(\mathbf{r}) \right)^2 + U(r) - E_F, \quad (2)$$

with $U(r)$ the potential due to a static distribution of impurities. The creation operators in the interaction term in Eq. (1) are understood to be superpositions of operators which create states only near the Fermi surface, that is, $c_{\mathbf{r}} = \int d\mathbf{p} \exp(i\mathbf{p} \cdot \mathbf{r}) c_{\mathbf{p}} \theta_{\mathbf{p}}$, where the theta-function limits the energy to be within a Debye frequency of the Fermi energy: $\theta_{\mathbf{p}} \equiv \theta(\omega_D - |\varepsilon_{\mathbf{p}} - E_F|)$. In the weak-coupling limit, where the range of the attractive potential is much smaller than the extent of a Cooper pair ($v_F/\omega_D \ll v_F/\Delta_0$), the interaction term may be considered local [32].

The variational approach to the Hamiltonian in Eq. (1) is defined by

$$\mathcal{H}' = \int d\mathbf{r} \Psi_{\mathbf{r}}^\dagger \begin{bmatrix} \mathcal{H}_0(\mathbf{r}) & \Phi(\mathbf{r}) \\ \Phi^*(\mathbf{r}) & -\mathcal{H}_0^*(\mathbf{r}) \end{bmatrix} \Psi_{\mathbf{r}}, \quad (3)$$

where

$$\Psi_{\mathbf{r}}^\dagger \equiv [c_{\mathbf{r}\uparrow}^\dagger \ c_{\mathbf{r}\downarrow}] .$$

The order parameter $\Phi(\mathbf{r})$ is determined self-consistently from the equation

$$\Phi(\mathbf{r}) = -V_0 \Omega \langle c_{\mathbf{r}\uparrow} c_{\mathbf{r}\downarrow} \rangle . \quad (4)$$

For a pure system ($U = 0$) with no magnetic field, plane waves diagonalize the bare Hamiltonian \mathcal{H}_0 . If the order parameter is assumed to be a constant, $\Phi(\mathbf{r}) = \Delta_0$, then Eq. (3) separates into 2×2 matrices. This yields the standard BCS results: the spectrum is given by $E_{\mathbf{k}} = \sqrt{\varepsilon_{\mathbf{k}}^2 + \Delta_0^2}$, where $\varepsilon_{\mathbf{k}} = k^2/2m - E_F$, and Eq. (4) becomes the BCS gap equation $\rho_0 V_0 \int d\varepsilon_{\mathbf{k}}/2E_{\mathbf{k}} = 1$, with ρ_0 the density of states at the Fermi level.

In the case of a system with impurities ($U \neq 0$), but with no magnetic field, a similar result holds, as shown by Anderson [17]. If we write the eigenstates of the bare Hamiltonian as

$$\mathcal{H}_0(H=0) \psi_\alpha^0 = \varepsilon_\alpha \psi_\alpha^0 \quad (5)$$

and rotate to a basis defined by these eigenstates,

$$c_{\alpha\sigma}^\dagger \equiv \int d\mathbf{r} \psi_\alpha^0(\mathbf{r}) c_{\mathbf{r}\sigma}^\dagger, \quad (6)$$

then the off-diagonal, pairing term in \mathcal{H}' becomes

$$\int d\mathbf{r} \Phi(\mathbf{r}) c_{\mathbf{r}\uparrow} c_{\mathbf{r}\downarrow} = \tilde{\Delta} \sum_{\alpha\beta} c_{\alpha\uparrow} c_{\beta\downarrow} \mathcal{A}_{\alpha\beta}^0, \quad (7)$$

where

$$\mathcal{A}_{\alpha\beta}^0 \equiv \int d\mathbf{r} \psi_\alpha^0(\mathbf{r}) \psi_\beta^0(\mathbf{r}). \quad (8)$$

Note that we explicitly assume that the order parameter is spatially homogeneous: $\Phi(\mathbf{r}) = \tilde{\Delta}$. The time-reversal symmetry of the system (no magnetic field, no magnetic impurities) implies that $\mathcal{H}_0 = \mathcal{H}_0^*$ and therefore a basis may be chosen in which $\psi_\alpha^0 = \psi_\alpha^{0*}$. The overlap integral in Eq. (8) then yields

$$\mathcal{A}_{\alpha\beta}^0 = \delta_{\alpha\beta} \quad (9)$$

The Hamiltonian \mathcal{H}' again factorizes into 2×2 matrices, the spectrum is again given by $E_\alpha = \sqrt{\varepsilon_\alpha^2 + \tilde{\Delta}^2}$ and solving the self-consistency equation yields the same result as in the impurity free case (assuming the density of states at the Fermi level is unchanged): $\Delta = \Delta_0$.

The random matrix description we will use for type-II superconductors [33] is a generalization of the Anderson description in the sense that we start by using exact eigenstates of the bare Hamiltonian \mathcal{H}_0 as a basis. This includes whatever vortex or impurity distribution is present. The eigenstates are

$$\mathcal{H}_0 \psi_\alpha = \varepsilon_\alpha \psi_\alpha, \quad (10)$$

and we rotate to the basis

$$c_{\alpha\sigma} \equiv \int d\mathbf{r} \psi_\alpha(\mathbf{r}) c_{\mathbf{r}\sigma}. \quad (11)$$

There are two important differences from the zero magnetic field case. First, the order parameter will no longer be spatially homogeneous; the phase winds rapidly throughout the superconductor (by 2π near the center of each vortex) and the magnitude vanishes at the center of each vortex. In order to extract a typical measure of the magnitude of the superconductivity, we define

$$\Phi(\mathbf{r}) \equiv \phi \chi(\mathbf{r}), \quad (12)$$

where $\chi(\mathbf{r})$ is normalized so that $\int |\chi(\mathbf{r})|^2 d\mathbf{r} = \Omega$. Hence ϕ^2 is the spatial average of the magnitude of the order parameter. Second, we no longer have time-reversal symmetry, because of the applied magnetic field. We still define the pairing matrix as

$$\mathcal{A}_{\alpha\beta} \equiv \int d\mathbf{r} \chi(\mathbf{r}) \psi_\alpha(\mathbf{r}) \psi_\beta(\mathbf{r}), \quad (13)$$

but we no longer have $\mathcal{A}_{\alpha\beta} = \delta_{\alpha\beta}$. Specifically, writing

$$\mathcal{A}_{\alpha\beta} = g_{\alpha\beta} h^{1/2}(\varepsilon_\alpha - \varepsilon_\beta), \quad (14)$$

then in zero magnetic field we have $h(\varepsilon_\alpha - \varepsilon_\beta) = \delta_{\alpha\beta}$. In the next section we will show that for a randomly disordered electron gas in a magnetic field, the δ -function

distribution for $h(\varepsilon)$ broadens by an amount proportional to the magnetic field.

The motivation behind the random matrix model is that $g_{\alpha\beta}$ is a material-specific, rapidly varying complex function, the details of which should not affect the density of states averaged over a random ensemble of disordered superconductors. The average density of states is sensitive to the overall structure of the pairing matrix elements – the broadening of the zero-field δ -function distribution of $h(\varepsilon)$ – but not to the rapidly varying $O(1)$ complex numbers which multiply it.

The basic assumption of the random matrix model is therefore that average properties of an ensemble of superconductors can be modeled by choosing complex numbers $g_{\alpha\beta}$ from a random, uncorrelated distribution:

$$\left[g_{\alpha\beta} g_{\alpha'\beta'}^* \right]_{\text{av}} = \delta_{\alpha\alpha'} \delta_{\beta\beta'} , \quad (15)$$

where the brackets denote an average over the distribution: $[X]_{\text{av}} \equiv (1/N_\mu) \sum_{\mu=1}^{N_\mu} X^{(\mu)}$. The particular choice of symmetry for $g_{\alpha\beta}$, whether real symmetric (GOE), Hermitian (GUE), or complex symmetric, will affect the eigenvalue spacing distribution but will not affect the density of states in the large matrix size limit.

Using the above definitions for $\Phi(\mathbf{r})$ and \mathcal{A} , and rotating to the basis of bare eigenstates defined by Eq. (11), the Hamiltonian in Eq. (3) becomes

$$\mathcal{H}' = \tilde{\Psi}^\dagger \begin{bmatrix} E_0 & \phi \mathcal{A} \\ \phi \mathcal{A}^\dagger & -E_0 \end{bmatrix} \tilde{\Psi} , \quad (16)$$

where $E_0 = \text{diag}(\dots \varepsilon_\alpha \dots)$ is the diagonal matrix of the eigenvalues of \mathcal{H}_0 , and $\tilde{\Psi}^\dagger = [\dots c_{\alpha\uparrow}^\dagger \dots \dots c_{\beta\downarrow} \dots]$. The assumption of uncorrelated random numbers for $g_{\alpha\beta}$ allows us to determine exactly, in the large matrix limit, the Green's functions of \mathcal{H}' [33].

III. THE PAIRING MATRIX AND THE COOPERON

The random matrix assumption for $g_{\alpha\beta}$ lets us write down integral equations for the Green's functions of the Hamiltonian \mathcal{H}' for any value of ϕ , the overall strength of the pairing, and $h(\omega)$, the average pairing amplitude defined in Eq. (14). This will only be useful in comparing with experiment if we have some prediction for, or theoretical understanding of, the pairing amplitude $h(\omega)$. For a diffusive system, and a disordered vortex lattice, there is a simple prediction.

We can rewrite Eq. (14) to express $h(\omega)$ in terms of the average squared matrix element as a function of energy difference:

$$h(\omega) = \frac{1}{N\rho_0} \left[\sum_{\alpha\beta} |\mathcal{A}_{\alpha\beta}|^2 \delta(\omega - \varepsilon_\alpha + \varepsilon_\beta) \right]_{\text{av}} , \quad (17)$$

where the brackets denote averaging over the random ensemble. The density of levels of the normal metal, $\rho_0 \equiv \rho(E_F)$, is assumed to be independent of energy in the narrow range about the Fermi energy relevant to superconductivity. The total number of levels that are being paired in \mathcal{H}' , which determines the size N of the matrix, is the level density times the maximum pairing energy. This cutoff is usually taken to be the Debye energy, so $N = \rho_0 \omega_D$. The normalization factor $1/N\rho_0$ in Eq. (17) follows from Eq. (14) and the continuous limit for the energy levels: $\sum_\alpha \rightarrow \int d\varepsilon_\alpha \rho(\varepsilon_\alpha)$.

Note also that we will always be interested in the limit where $N \rightarrow \infty$, because ω_D is much larger than the level spacing (as well as the pairing strength ϕ and the bare BCS gap Δ_0). In order for the Hamiltonian \mathcal{H}' in Eq. (16) to be interesting in that limit, the matrix elements $\mathcal{A}_{\alpha\beta}$ must be $O(1/\sqrt{N})$. (This is easy to see with perturbation theory: the shift in energy of some level ε_α is $\delta\varepsilon_\alpha = \phi^2 \sum_\beta |\mathcal{A}_{\alpha\beta}|^2 / (\varepsilon_\alpha + \varepsilon_\beta)$. The N terms in the sum must be canceled by a $1/N$ factor from $|\mathcal{A}_{\alpha\beta}|^2$.)

Inserting Eq. (13) for $\mathcal{A}_{\alpha\beta}$ into Eq. (17) yields

$$h(\omega) = \frac{1}{N\rho_0} \left[\sum_{\alpha\beta} \int d\mathbf{r} d\mathbf{r}' \psi_\alpha(\mathbf{r}) \psi_\beta(\mathbf{r}) \psi_\alpha^*(\mathbf{r}') \psi_\beta^*(\mathbf{r}') \right. \\ \left. \times \chi(\mathbf{r}) \chi^*(\mathbf{r}') \delta(\omega - \varepsilon_\alpha + \varepsilon_\beta) \right]_{\text{av}} . \quad (18)$$

We can insert a factor of $1 = \int dE \delta(E - \varepsilon_\alpha)$ and write this as

$$h(\omega) = \frac{(2\pi)^{-2}}{N\rho_0} \int d\mathbf{r} d\mathbf{r}' dE \left[\delta G_0(\mathbf{r}, \mathbf{r}', E) \right. \\ \left. \times \delta G_0(\mathbf{r}, \mathbf{r}', E + \omega) \chi(\mathbf{r}) \chi^*(\mathbf{r}') \right]_{\text{av}} , \quad (19)$$

where the bare single particle Green's functions are

$$\delta G_0 \equiv G_0^+ - G_0^- \\ G_0^\pm(\mathbf{r}, \mathbf{r}', E) \equiv \sum_\alpha \frac{\psi_\alpha(\mathbf{r}) \psi_\alpha^*(\mathbf{r}')}{E - \varepsilon_\alpha \pm i0^+} .$$

To make further progress, we will need to make a key assumption: that there is, on average, no correlation between the bare single particle Green's functions G_0^\pm and the order parameter product $\chi(\mathbf{r})\chi^*(\mathbf{r}')$ appearing above:

$$\left[G_0^\pm G_0^\pm \chi \chi^* \right]_{\text{av}} = \left[G_0^\pm G_0^\pm \right]_{\text{av}} \left[\chi \chi^* \right]_{\text{av}} . \quad (20)$$

The wavefunctions which enter G_0 are the eigenstates of the bare Hamiltonian \mathcal{H}_0 , not the eigenstates of the full superconducting Hamiltonian. Hence the assumption is that the product of the bare Green's functions is uncorrelated, on average, with the order parameter, that is, the positions of the vortices. Below, we will only be interested in the case in which the magnetic field is nearly uniform within the superconductor, $H \gg H_{c1}$. In this limit, since the magnetic field in \mathcal{H}_0 is uniform, the positions

of the vortices affect G_0 only through a gauge-dependent phase. The quantity we are interested in, $h(\omega)$, is gauge-invariant. Therefore, with the gauge-invariance of the product of the two terms properly maintained, and for $H \gg H_{c1}$, there are no additional correlations between the bare Green's functions and the order parameter.

With the average over the order parameter product written as

$$\left[\chi(\mathbf{r}) \chi^*(\mathbf{r}') \right]_{\text{av}} \equiv g(\mathbf{r}, \mathbf{r}') , \quad (21)$$

the expression for $h(\omega)$ then becomes

$$h(\omega) = \frac{(2\pi)^{-2}}{N\rho_0} \text{Re} \int d\mathbf{r} d\mathbf{r}' \left[\int dE G_0^+(\mathbf{r}, \mathbf{r}', E) \right. \\ \left. \times G_0^-(\mathbf{r}, \mathbf{r}', E + \omega) \right]_{\text{av}} g(\mathbf{r}, \mathbf{r}') , \quad (22)$$

where we have used the identity

$$\left(G_0^+(\mathbf{r}, \mathbf{r}', E) G_0^-(\mathbf{r}, \mathbf{r}', E + \omega) \right)^* = \\ G_0^-(\mathbf{r}', \mathbf{r}, E) G_0^+(\mathbf{r}', \mathbf{r}, E + \omega) \quad (23)$$

and we omit the terms involving G^-G^- and G^+G^+ because they do not contribute.

As mentioned in the introduction, in weak localization theory quantum corrections to classical transport are calculated by studying averaged values of response functions of non-interacting electrons in a random disorder potential. The quantity in brackets in Eq. (22) is one such disorder-averaged response function: the Cooperon [29,30].

The order of \mathbf{r} and \mathbf{r}' in Eq. (22) is important. If the single particle Green's functions appeared in the form $G_0(\mathbf{r}, \mathbf{r}', E) G_0(\mathbf{r}', \mathbf{r}, E + \omega)$, the quantity in brackets would be just the real-space Fourier transform of the usual density-density response function. It is because $\mathcal{A}_{\alpha\beta}$ is a matrix element for electrons in a Cooper pair (the overlap between ψ_α and ψ_β rather than ψ_α^* and ψ_β), that the order of \mathbf{r} and \mathbf{r}' is reversed, and the bracketed quantity in Eq. (22) is the Cooperon rather than the diffuson.

The Cooperon response function $C(\mathbf{r}, \mathbf{r}', \omega)$ is defined as

$$C(\mathbf{r}, \mathbf{r}', \omega) = \left[\int dE G_0^+(\mathbf{r}, \mathbf{r}', E) G_0^-(\mathbf{r}, \mathbf{r}', E + \omega) \right]_{\text{av}} . \quad (24)$$

This response function, in the limit in which the mean free path is much smaller than the magnetic length but much larger than the Fermi wavelength ($\lambda_F \ll \ell \ll \ell_H \equiv \sqrt{\hbar c / eH}$), obeys a diffusive equation of motion [29]

$$\left[-i\omega + D \left(i\nabla + \frac{2e}{\hbar c} \mathbf{A} \right)^2 \right] C(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r} - \mathbf{r}') . \quad (25)$$

The diffusion constant $D = v_F \ell / d$, where d is the space dimensionality. This equation defines the Green's function for the Schroedinger equation of a single particle of charge $2e$ in a magnetic field. The solution is therefore

$$C(\mathbf{r}, \mathbf{r}', \omega) = \sum_{nk} \int \frac{dk_z}{2\pi} \frac{\varphi_{nk}(x, y) \varphi_{nk}^*(x', y') e^{ik_z(z-z')}}{-i\omega + (4D/\ell_H^2)(n + 1/2) + Dk_z^2} , \quad (26)$$

where $\varphi_{nk}(x, y)$ are normalized 2D Landau level wavefunctions and we assume a uniform magnetic field H (valid for $H \gg H_{c1}$).

This solution is not sufficient to evaluate $h(\omega)$; we also need to know the order parameter correlation, $g(\mathbf{r}, \mathbf{r}')$, defined by Eqs. (12) and (21). The phase of the order parameter winds by 2π around each vortex. In a superconductor with a perfectly periodic vortex lattice we expect $\Phi(\mathbf{r})\Phi^*(\mathbf{r}')$ to reflect the long-range ordered structure of the vortex lattice. Averaged over an ensemble of disordered materials, however, the phase will disorder on the length scale set by the inter-vortex spacing ℓ_H . Therefore we will assume that the averaged correlation function has the form

$$g(\mathbf{r}, \mathbf{r}') = e^{-|\mathbf{r}-\mathbf{r}'|^2/\ell_H^2} e^{i\theta_{\text{gauge}}(\mathbf{r}, \mathbf{r}')} . \quad (27)$$

The gauge-dependent phase factor is required to keep the product $C(\mathbf{r}, \mathbf{r}', \omega)g(\mathbf{r}, \mathbf{r}')$ gauge-invariant. In the Landau gauge $\mathbf{A} = Hx\hat{y}$, we have $\theta_{\text{gauge}}(\mathbf{r}, \mathbf{r}') = (x + x')(y - y')/2\ell_H^2$.

The form of $|g(\mathbf{r}, \mathbf{r}')|$ is an additional assumption that is necessary for evaluating $h(\omega)$. It is not essential that this form be Gaussian. However, the Gaussian form is natural for a random system, and also has the property that the result for $h(\omega)$ is particularly simple.

We now substitute the expressions for $C(\mathbf{r}, \mathbf{r}', \omega)$ and $g(\mathbf{r}, \mathbf{r}')$, Eqs. (26) and (27), into Eq. (22). The integral over $z - z'$ sets $k_z = 0$ in the denominator. The Gaussian form for $g(\mathbf{r}, \mathbf{r}')$ can be written as $|g| = \sqrt{\pi} h_0((x - x')/\ell_H) h_0((y - y')/\ell_H)$, where $h_n(u)$ are the normalized harmonic oscillator eigenstates (Hermite functions). The 2D Landau level wavefunctions in Landau gauge have the form $\varphi_{nk}(x, y) = A \exp(ikx) h_n(k\ell_H/\sqrt{2} + \sqrt{2}y/\ell_H)$. Rewriting the product in Eq. (26) using the identity $h_n(a)h_n(b) = \sum_j C_{nn}^j h_{2n-j}((a+b)/\sqrt{2}) h_j((a-b)/\sqrt{2})$ [34], we see that the spatial integrals in Eq. (22) and the orthogonality of the Hermite functions set $n = 0$. The final result is therefore that only the vanishing n and k_z terms survive in the sums in Eq. (26). This yields for $h(\omega)$:

$$h(\omega) = \frac{1}{2\pi\rho_0} \frac{W}{\omega^2 + W^2} , \quad (28)$$

where the width W is

$$W = 2\hbar D/\ell_H^2 = 2eDH/c , \quad (29)$$

and we have normalized to $\rho_0 \int d\omega h(\omega) = 1$.

The Lorentzian form for $h(\omega)$, Eq. (28), is what might be expected on general grounds in a diffusive system where the time-evolution of pair correlations obeys an exponential decay law. This point was discussed by de Gennes in the context of the “ergodic evolution of the time-reversal operator” [35]. de Gennes considered the problem of evaluating the transition temperature $T_c(H)$ for a general superconducting system in a magnetic field described by the variational Hamiltonian in Eq. (3). He used a perturbative expansion in the order parameter $\Phi(\mathbf{r})$, which is valid near the phase transition, and found that the average over single-particle wavefunctions which appears in Eq. (18) is the kernel operator acting on $\Phi(\mathbf{r})$ in the gap equation. de Gennes related that average over single-particle wavefunctions to the time evolution of the time-reversal operator, $K(t)$, for one electron moving in the potential described by the Hamiltonian \mathcal{H}_0 (Eq. (2)). An “ergodic” system was defined as one in which the motion of the electron is diffusive, in the sense that the time-reversal operator decays exponentially with time:

$$\lim_{t \rightarrow \infty} \langle K^\dagger(0)K(t) \rangle = e^{-t/\tau} . \quad (30)$$

For a system defined as ergodic in this sense, $h(\omega)$, which is the power spectrum of the operator $K(t)$, has the Lorentzian form given in Eq. (28), with $W = \tau^{-1}$.

The interesting result here is that the average over the single-particle wavefunctions in Eq. (18) appears not only in the equation for $T_c(H)$, but also in the random matrix description of the pairing Hamiltonian. The functional form of $h(\omega)$ and the random matrix technique allow us to go beyond perturbation theory and therefore beyond calculating $T_c(H)$. At any field for which the random matrix element hypothesis is valid, we can obtain the full Green’s functions non-perturbatively.

We note also that the connection with “ergodic superconductors” suggests that the random matrix description may apply, in addition to the bulk disordered systems considered here, to other systems discussed by de Gennes *et al.* as being ergodic, for example, thin films and superconducting nanoparticles [35,36].

IV. UPPER CRITICAL FIELD

Using Eqs. (28) and (29) we may derive a relation between the upper critical field H_{c2} and the diffusion constant D . This will allow us to test whether the formalism described in the previous section yields a result consistent with the prediction of the semiclassical theory of type-II superconductors. That theory is based on the perturbative solution of the Gor’kov equations near H_{c2} using semiclassical Green’s functions to include the effects of the magnetic field.

The difference in energy between the normal metal ground state of the Hamiltonian in Eq. (3), given by $\phi = 0$, and the superconducting ground state, is, to second order in ϕ :

$$\Delta E = \phi^2 \left[\frac{1}{V_0} - \sum_{\alpha\beta} \frac{|\mathcal{A}_{\alpha\beta}|^2}{|\varepsilon_\alpha + \varepsilon_\beta|} \theta(\varepsilon_\alpha \varepsilon_\beta) \right] + O(\phi^4) . \quad (31)$$

The $|\mathcal{A}_{\alpha\beta}|^2$ term is second order perturbation theory for the Hamiltonian Eq. (3), where $\theta(x)$ is the step function (for non-zero temperatures, $\theta(\varepsilon_\alpha \varepsilon_\beta)$ is replaced by $[\tanh(\varepsilon_\alpha/2T) + \tanh(\varepsilon_\beta/2T)]/2$) and the upper cutoffs for the energies are $|\varepsilon_\alpha| < \omega_D$, $|\varepsilon_\beta| < \omega_D$. The $1/V_0$ term is the usual Hartree-Fock self-consistency contribution and follows from $\langle \int d\mathbf{r} \Phi(\mathbf{r}) c_{\mathbf{r}\uparrow} c_{\mathbf{r}\downarrow} \rangle$ and $\Phi(\mathbf{r}) = -\Omega V_0 \langle c_{\mathbf{r}\uparrow} c_{\mathbf{r}\downarrow} \rangle$. We must ignore small non-perturbative effects [37] (the breakdown of second-order perturbation theory), in order to reproduce the semiclassical limit.

The upper critical field is the field H_{c2} at which the quantity in brackets in Eq. (31) vanishes. At lower fields, the coefficient of ϕ^2 is negative and $\phi > 0$ minimizes the energy; at higher fields, the coefficient of ϕ^2 is positive and $\phi = 0$ minimizes the energy.

Averaging over the ensemble amounts to replacing $|\mathcal{A}_{\alpha\beta}|^2$ in Eq. (31) by its average value $h(\varepsilon_\alpha - \varepsilon_\beta)$. Taking the continuum limit for the energies, inserting Eq. (28) for $h(\nu)$, and using the definition of the BCS coupling constant, $1/g \equiv 1/\rho_0 V_0$, yields that the $O(\phi^2)$ term in Eq. (31) vanishes when

$$\frac{1}{g} = \frac{2}{\pi} \int_0^{\omega_D} \int_0^{\omega_D} \frac{d\varepsilon_\alpha d\varepsilon_\beta}{\varepsilon_\alpha + \varepsilon_\beta} \frac{W}{(\varepsilon_\alpha - \varepsilon_\beta)^2 + W^2} . \quad (32)$$

Performing the integrals yields

$$\frac{1}{g} = \log \left(\frac{2\omega_D}{W} \right) \left[1 + O \left(\frac{W}{\omega_D} \right) \right] . \quad (33)$$

In the weak coupling limit $W/\omega_D \rightarrow 0$. Subtracting the zero-field BCS result,

$$\frac{1}{g} = \log \left(\frac{2\omega_D}{\Delta_0} \right) , \quad (34)$$

where Δ_0 is the zero-field BCS gap, we obtain the equation which determines the upper critical field:

$$W(H_{c2}) = \Delta_0 . \quad (35)$$

Inserting Eq. (29) for $W(H)$ yields

$$H_{c2} = \frac{3}{2\pi^2} \frac{\Phi_0}{\ell \xi_0} , \quad (36)$$

where $\xi_0 = \hbar v_F / \pi \Delta_0$ is the zero-field BCS coherence length and $D = v_F \ell / 3$ in three dimensions.

This result may be compared with the result derived with the semiclassical theory of dirty type-II superconductors. The semiclassical approach starts with plane

wave electronic eigenstates, assumes a weak magnetic field, and includes the effects of impurities through the Abrikosov-Gor'kov impurity averaged perturbation theory technique. The resulting Gor'kov equations can be expanded perturbatively in the order parameter (near H_{c2}) and yield the equation which determines H_{c2} [38–40]

$$\ln\left(\frac{T_{c0}}{T}\right) = -\psi\left(\frac{1}{2}\right) + \psi\left(\frac{1}{2} + \frac{H_{c2}^d D}{2\Phi_0 T}\right), \quad (37)$$

where $\psi(x)$ is the digamma function, T_{c0} is the zero-field critical temperature, and H_{c2}^d denotes the dirty limit, $\ell \ll \xi_0$. As $T \rightarrow 0$ this gives $H_{c2}^d = (3/2\pi^2)(\Phi_0/\ell\xi_0)$, in agreement with Eq. (36).

The agreement with the semiclassical theory shows that the random matrix description combined with the results from the Cooperon problem reproduces the perturbative result of the more conventional approach to understanding dirty type-II superconductors.

V. CONNECTION WITH ABRIKOSOV-GOR'KOV THEORY, AND THE $H \rightarrow 0$ LIMIT

As pointed out in Ref. [33], in the weak field limit, for which the T-breaking scale $W \ll \Delta_0$, the random matrix model described in the previous sections does *not* yield the density of states the zero-field BCS theory. What happens in this limit is that the central assumption of the model, that the Hamiltonian of the superconductor can be well described by a large matrix with random pairing elements, breaks down. For small W/Δ_0 there exists a special basis, discussed below, in which the Hamiltonian has a simpler form. The existence of a special basis means that correlations have developed which are not obvious in the original basis. (As a simple analogy, consider an electron in a cylindrically symmetric potential. Random matrix elements in a plane wave basis would be a poor assumption.)

To understand what happens to the random matrix model of the previous sections in the weak field limit, we first write down two additional possible random matrix models for superconductors with T-breaking. The first of these turns out to be a random matrix formulation of the Abrikosov-Gor'kov theory. The second of these we will argue describes the weak field limit, $H \ll H_{c2}$, of a superconductor in a magnetic field.

The theory described in the previous sections we denote by H_I :

$$H_I = \begin{bmatrix} E_0 & \phi \mathcal{A} \\ \phi \mathcal{A}^\dagger & -E_0 \end{bmatrix}, \quad (38)$$

with \mathcal{A} satisfying

$$[\mathcal{A}_{ij} \mathcal{A}_{kl}^*]_{\text{av}} = \delta_{ik} \delta_{jl} h(\varepsilon_i - \varepsilon_j).$$

As before, $E_0 = \text{diag}(\dots \varepsilon_\alpha \dots)$ is a diagonal matrix of uniformly distributed eigenvalues.

A different possible random matrix model occurs if we start with BCS and add a random matrix to the diagonal components, in a way which breaks the time-reversal symmetry:

$$H_{II} = \begin{bmatrix} E_0 & \Delta \\ \Delta & -E_0 \end{bmatrix} + \alpha \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}, \quad (39)$$

with M satisfying

$$[M_{ij} M_{kl}^*]_{\text{av}} = \delta_{ik} \delta_{jl}.$$

E_0 is a diagonal matrix as in the previous case, Δ is proportional to the identity matrix: $\Delta = \Delta_0 \delta_{ij}$, α is a parameter which measures the amount of T-breaking, and M is a random Hermitian matrix with *no* weighting factor $h(\varepsilon_i - \varepsilon_j)$ in it. The T-breaking in this model is represented by the same sign for M in the upper and lower diagonal quadrants. Since these quadrants correspond to the energies for up-spin electrons and down-spin holes, this indicates a potential which has opposite sign for up and down spins. In other words, this model appears to correspond to random magnetic impurities. In fact, the equations for the Green's functions of this random matrix model are the same as the Abrikosov-Gor'kov equations for a superconductor in the presence of random magnetic impurities. This random matrix model is a reformulation of the AG theory [41].

The final random matrix model we consider is

$$H_{III} = \begin{bmatrix} E_0 & \Delta \\ \Delta & -E_0 \end{bmatrix} + \alpha_1 \begin{bmatrix} M^{(1)} & 0 \\ 0 & M^{(1)} \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & M^{(2)} \\ M^{(2)} & 0 \end{bmatrix}, \quad (40)$$

with the matrices $M^{(a)}$ satisfying

$$[M_{ij}^{(a)} M_{kl}^{(b)*}]_{\text{av}} = \delta_{ab} \delta_{ik} \delta_{jl}.$$

This differs from H_{II} in that a random matrix appears in both the diagonal and off-diagonal quadrants.

How do we best describe a superconductor in the weak field limit, $H \ll H_{c2}$? The order parameter in a superconductor may be written as

$$\Phi(\mathbf{r}) = f(\mathbf{r}) e^{i\theta(\mathbf{r})}, \quad (41)$$

where $f(\mathbf{r})$ is real and $\theta(\mathbf{r})$ is a position dependent phase that winds by 2π around the center of each vortex. For a given material, with some order parameter phase realization $\theta(\mathbf{r})$, we may transform the standard creation operators to a new basis defined by

$$c_{\mathbf{r}} \rightarrow \tilde{c}_{\mathbf{r}} \equiv c_{\mathbf{r}} e^{i\theta(\mathbf{r})/2}. \quad (42)$$

In this basis the pairing interaction takes the form

$$\Phi(\mathbf{r}) c_{\mathbf{r}\uparrow}^\dagger c_{\mathbf{r}\downarrow}^\dagger \rightarrow f(\mathbf{r}) \tilde{c}_{\mathbf{r}\uparrow}^\dagger \tilde{c}_{\mathbf{r}\downarrow}^\dagger. \quad (43)$$

The effective order parameter in the new basis is purely real. Note that although this has sometimes been called a gauge transformation, it is not: the magnetic field changes. The vector potential is $\tilde{\mathbf{A}}(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla\theta/2e$ so $\int \tilde{\mathbf{A}}(\mathbf{r}) \cdot d\mathbf{l} \neq \int \mathbf{A}(\mathbf{r}) \cdot d\mathbf{l}$ around a loop enclosing a vortex.

Physically, we are transforming to a basis in which the magnetic field is the previous field distribution plus one flux quantum threaded through the center of each vortex in a direction opposite to that of the applied field. This means that far away from an isolated vortex, the net effective flux seen by an electron is zero rather than one flux quantum. In the original basis, the vector potential \mathbf{A} falls off as $1/r$ for distances greater than a penetration depth away from a single vortex. The transformed vector potential, $\tilde{\mathbf{A}}$, follows the superfluid velocity $\mathbf{v}_s(\mathbf{r})$, and vanishes exponentially for distances greater than a penetration depth.

The matrix model of the previous sections, H_I , uses as a basis the eigenstates of the full Hamiltonian \mathcal{H}_0 , which includes the magnetic field. The models described by H_{II} and H_{III} first apply the above transformation, then use the basis defined by the eigenstates of just the T-invariant part of the Hamiltonian. This has two important effects when the magnetic field is weak.

First, weak T-breaking implies that the phase gradient of the order parameter varies slowly on the scale of a coherence length: $\xi_0 \nabla\theta \ll 2\pi$. Therefore the term in the bare Hamiltonian due to the superfluid velocity is small compared to the energy gap: $(1/m)\mathbf{p} \cdot e\tilde{\mathbf{A}}(\mathbf{r}) \sim v_F \nabla\theta \ll \Delta_0$. Second, weak T-breaking in a conventional superconductor means that the deviation in the magnitude of the order parameter from the bare BCS value is small: $\delta f/\Delta_0 \equiv \int (d\mathbf{r}/\Omega) |f(\mathbf{r}) - \Delta_0|/\Delta_0 \ll 1$. In the Ginzburg-Landau description of an isolated vortex, the order parameter relaxes to the bare value exponentially, on the scale of the coherence length. Hence $\delta f/\Delta_0 \sim \xi_0^2/\ell_H^2 = H/H_{c2}$. In a more realistic description, $f(\mathbf{r})$ may have power-law behavior $|f(\mathbf{r}) - \Delta_0| \sim (\xi_0/r)^n$ which yields $\delta f/\Delta_0 \sim (H/H_{c2})^{1/2}$ ($n = 1$), or $\delta f/\Delta_0 \sim (H/H_{c2}) \ln(H_{c2}/H)$ ($n = 2$). In either case, rotating to a basis of the eigenstates of the T-invariant part of \mathcal{H}_0 for which the order parameter is real yields a Hamiltonian which is a small correction to the BCS theory.

The models H_{II} and H_{III} are defined by the assumption that the matrix elements of the T-breaking corrections to the BCS theory are uncorrelated. That is, in Eqs. (39) and (40), the matrices M and $M^{(1)}$ (the superfluid velocity correction), and $M^{(2)}$ (the order parameter correction), are assumed to be random. The difference between the two models is that in H_{II} the effect of the deviation of the order parameter from the constant magnitude is ignored, whereas in H_{III} it is included. Hence in H_{II} the pairing is diagonal, $\Delta_0 \delta_{ij}$, because $\langle i|T|j \rangle = \delta_{ij}$

(T is the time reversal operator $T : \psi_i(\mathbf{r}) \rightarrow \psi_i^*(\mathbf{r})$), while in H_{III} , the deviation of the order parameter yields an additional correction: $\langle i|Tf(\mathbf{r})|j \rangle = \Delta_0 \delta_{ij} + \alpha_2 M_{ij}^{(2)}$, where $\alpha_2 \sim \delta f$.

These models describe small corrections to BCS and hence properly reproduce the BCS result as $H/H_{c2} \rightarrow 0$. The matrix model of the previous sections, H_I , can not be described as a small correction to BCS. In the weak field limit, $W/\Delta_0 \sim H/H_{c2} \rightarrow 0$, so $h(\omega) \rightarrow \delta(\omega)$. However, because the bare level spacing is always assumed to be much less than the T-breaking scale ($\delta_0 \ll W$), the resulting matrix model still pairs many levels with equal, random amplitude. This leads to large deviations from BCS. For $W/\Delta_0 \rightarrow 0$, the density of states has a node near zero energy, $\rho(E) \sim E$, rather than a gap as in the BCS spectrum [33].

At weak fields, or more generally for weak T-breaking, when the correction to BCS behavior can be considered small, the most appropriate random matrix description is that of the third model listed above: H_{III} . As the T-breaking energy scale increases, this description must break down. Although the order parameter can always be written as the BCS value plus a correction, $f(\mathbf{r}) = \Delta_0 + \delta f(\mathbf{r})$, the assumption that the matrix element $\langle i|T\delta f(\mathbf{r})|j \rangle$ is random is no longer justifiable when $\delta f/\Delta_0 \sim 1$; $f(\mathbf{r})$ and $\delta f(\mathbf{r})$ are comparable functions. At sufficiently high fields, the only consistent assumption of randomness applies to the entire pairing matrix, as assumed in H_I . This is called ‘chaotic’ pairing.

Figure 1 illustrates schematically the different regimes.

Physically, at low fields, vortices are well separated, bound states are confined primarily to individual vortices, and the effect of the superfluid velocity on extended states is small, yielding only a small correction to the ordered BCS state. At higher fields, core states overlap significantly and form energy bands, which interact in a complicated way with extended states, the superfluid velocity, and whatever disorder is present. In the Anderson metal-insulator transition, delocalization is accompanied by a change from ordered (Poisson) statistics to chaotic (random matrix) statistics. In a type-II superconductor, we are suggesting that the delocalization due to increasing vortex density (increasing magnetic field) is accompanied by a transition from a small correction to the BCS theory to chaotic pairing. This raises interesting possibilities and questions, discussed in the next section, regarding the nature of such a transition in a superconductor.

VI. SUMMARY AND CONCLUSIONS

When the disorder and magnetic field are sufficiently large in a type-II superconductor, so that electronic structure calculations and small deviations from the BCS

theory are no longer applicable, the chaotic pairing picture of the random matrix model may apply. The motivation for the model is that the density of states is not sensitive to details of the rapidly varying $O(1)$ factors in the matrix elements between electrons in a Cooper pair, but rather to the average structure of the pairing matrix.

In section III we showed that the average amplitude to pair electrons is related to the Cooperon response function previously discussed in the context of weak localization theory. For a dirty metal this response function may be calculated and yields a simple form for the average pairing amplitude, characteristic of a diffusive system.

The connection between the average pairing amplitude and diffusion is related to what was called by de Gennes “ergodic pairing”. That connection was established by de Gennes in the context of evaluating the critical field of dirty superconductors, H_{c2} , using perturbation theory. Here we have shown that this connection exists independent of perturbation theory. This allows us to evaluate the density of states for fields far away from the regime in which perturbation theory is valid.

Finally, we considered the weak field limit of the random matrix model, $H \ll H_{c2}$. At sufficiently weak fields the random matrix picture must break down because the system must approach the ordered BCS state. This suggests a crossover between a low-field ordered state, in which the corrections to BCS theory are small and can be understood as being due to states localized near individual vortices, to a high-field chaotic state, in which most of the states are extended. A useful analogy is the Anderson metal-insulator transition. Instead of varying the electron density (chemical potential) from a regime of localized states to a regime of delocalized states, in the superconductor we vary the vortex density (magnetic field) between a regime of bound states to a regime of extended states. In the metal-insulator transition we have a change in statistics from Poisson to random matrices; in the superconductor we have a change in the pairing, from BCS-like to chaotic.

The connection between the field-dependence of the pairing in a superconductor and the nature of the low-energy quasiparticles raises several interesting questions. Is there a true localization phase transition for quasiparticles? Is there a sharp change between an ordered regime and a chaotic regime or merely a crossover? Are there simple models for which, using techniques similar to those in the theory of disordered metals, the chaotic pairing picture may be derived? What is the interplay between localization, disorder, and magnetic field for high- T_c superconductors, where the prediction in zero field for a two-dimensional d -wave superconductor is “universal conductivity” [42]? A further important issue is to test the validity of the random matrix prediction for the density of states, for each of the possible regimes, by direct comparison with tunneling experiments.

This work was supported by a Miller Post-Doctoral

Fellowship from the Miller Institute for Basic Research in Science. I would like to thank D.-H. Lee for additional support and useful discussions.

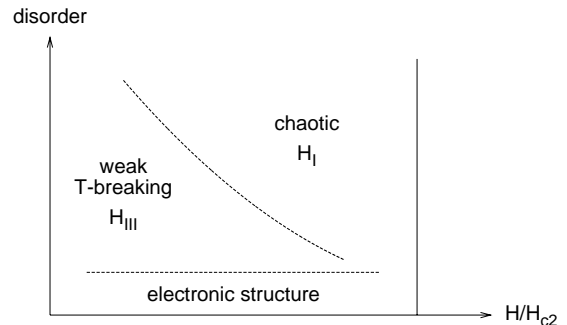


FIG. 1. In the absence of impurities and for an ordered lattice of vortices, the quasiparticle spectrum of a type-II superconductor may be calculated by numerically solving the Bogoliubov-de Gennes equations. In the presence of random disorder, and for weak magnetic fields, the quasiparticle spectrum may be calculated as a small correction to the BCS theory, Eq. (40), similar to the Abrikosov-Gor’kov theory. For stronger disorder or magnetic fields, the system enters a chaotic regime in which the pairing is completely random, Eq. (38).

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